Condensed excerpt. *Elements of Structured Finance*, Ch. 22: The Valuation of Structured Securities

"Μελέτα τό πάν" -Periander

I. Introduction: To *Value* a deal means to give it ground, and nothing else.

In assigning ratings to structured securities, Spectrum pays attention to *what* is really measured when a deal is valued, as opposed to *how* it is actually measured. The credit rating must reflect changes in value as driven by credit factors, or else the rating becomes decoupled from value and becomes a source of noise that can destabilize credit markets.

The basic problem of financial valuation (and hence rating financial assets) is to carry out the valuation process from the ground up in a self-consistent manner. If done correctly, price coalesces around *Value* in an obvious manner to enable *risk management*—which encompasses trading and own-risk—at the security's *equilibrium* price, aka *fair market value*. This process is precisely and exclusively how a market can develop for an asset. Valuation is how the deal comes together *as* a deal, as a manifold unity.

In ***Elements-Chapter 14***, we introduce non-linearity by showing how the deal comes together inside a feedback loop in yield space. Thus, for lack of a better word, yield is redefined as *original time*, the transcendental horizon for the understanding of dealing, where the deal emerges out of itself and yields its *Value*, its true price. By linking interest rates to themselves within a cybernetic feedback loop, they come to their final, *unique* resting place on their own. When it exists, this fixed point is the "*Value*" of the deal. What gives *Value* credibility and elevates it above the rank of opinion is not that it is cheap or rich, high or low, or reflects a pre-established, purported fount of human wisdom; but rather, it is grounded: *unique*.

II. The Structured Valuation Problem: Resolving Nonlinearity

A deal's *Value* is the limit point of an iterative procedure operating in a multi-dimensional, non-linear space. Its dimensionality equals to number of deal credit tranches. A two-tranche deal is a two-dimensional Euclidean space: the yield on any fixed income security can be regarded as a real number $r \in [-1,1]$.

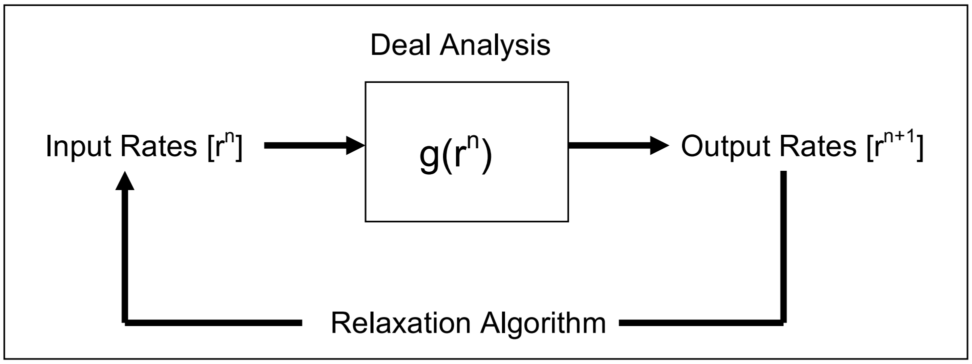
Spectrum values structured securities by exploring the dependencies of rates and ratings on the average reduction of yield a security-holder would experience over a range of realizations in Monte Carlo [MC] simulation. We say the results relate to the transaction under review by the ergodic hypothesis.

But, to know the rating requires knowledge of the rate; and to know the rate requires knowledge of the rating! We cannot redefine credit ratings as knowable *a priori*. The problem will simply reappear from another angle. The problem cannot be removed *via* mapping or transformation. This is the *non-linearity* of structured finance, which Spectrum resolves by the Non-Linear Convergence Algorithm:

1- Estimate an initial, provisional yield vector—one rate for each credit tranche in the deal.
2- Perform a first MC simulation using this provisional yield vector and other deal assumptions, and derive tranche credit ratings as their average yield reductions.
3- Use empirically derived yield-spread curves (or a model) to compute interest rates from both the average reduction of yield and the average life of each security.
4- Using the below relaxation method, transform the output rates into a new input-rate vector and substitute the latter into the MC engine.

5- Compute the absolute value difference between input and output rates expressed as a percentage of the input rates.
6- Define the convergence parameter δ as the percentage in (5) weighted by initial tranche balances.
7- Repeat the sequence above until δ falls below a specified error bound in the neighborhood of 1%.
8- The final yield vector can now be defined as the *Value* of the deal. Because *Value* is fair, the offered price of each security should be par.

Figure 22.1: The Valuation Schema in Structured Finance



The vector-valued mapping function $g(r)$ in Figure 22.1 stands for the complete deal cycle consisting of an entire MC simulation. Each simulation covers thousands of deal realizations, involving perhaps three hundred time steps each, and potentially thousands of obligors processed in each MC scenario. Yield curve modeling is involved in this process, as is credit-spread calculation. The rate relaxation algorithm can usually be subsumed under the mapping function $g(r)$ without loss of generality.

The iterative valuation scheme can be represented as $r^{n+1} = g(r^n)$ 22.5

It is now trivial to define theoretically the *Value* of the deal V as the fixed point r^∞ of the map $g(r)$:

$$V \equiv r^\infty = g(r^\infty) \qquad 22.6$$

The practical problem now becomes the determination of conditions under which the sequence of rate-vectors r^n will converge to some fixed limit. The behavior of non-linear iteration is idiosyncratic. Well-posed linear systems can almost always be solved, but non-linear maps present special problems. Their resolution requires a unique solution.

III. Contraction Mapping and Deal Valuation

Spectrum resolves this nonlinear iteration problem by the Banach contraction-mapping theorem. We posit it as the fundamental valuation theorem in structured finance.

<u>The Banach Contraction Mapping Theorem</u>

Let X be a complete[1] metric space equipped with metric d and let the map $g(x)$ be given by $g(x): X \to X$, mapping every element of X onto another element of X. The elements of X are conceived as vectors, i.e. groups of numbers with as many dimensions as we like.
This mapping is also a *contraction* mapping when any two elements of the domain are mapped to a pair of images that are closer to each other than the original pair of elements, i.e. the following condition holds for any choice of elements x and y in X for some $q, 0 \le q < 1$:

[1] A metric space M is complete if every Cauchy sequence in M converges in M.

$$d(g(x), g(y)) \le q\, d(x, y) \qquad 22.7$$

In one dimension the metric or *norm* is generally the distance between two elements and the absolute value of their algebraic difference. E.g. $|x - y|$, with each element represented by $x \equiv [x_1, x_2]$, where the metric is the distance between elements x and y using the Euclidean norm:

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \qquad 22.8$$

The metric can be generalized to more than two dimensions. Each dimension must be considered in turn and some other form must be used for the norm. For structured finance, we will generally need as many dimensions as there are credit tranches in the deal. RMBS have many liquidity tranches but usually no more than three credit tranches; ABS deals may have more.

By Theorem 1, if a deal's mapping $g(r)$ is contracting over some domain, a valuation exists as a unique fixed point. Sensitivity of the convergence (*basis of attraction*) increases with the number of tranches.

Theorem 1 (Banach):

For constant $q, 0 \le q \le 1$, every contaction mapping in X with metric d has a unique fixed point.

What this means in deal terms is that the distance between the initial EDR estimate and the actual EDR, always ideally small, must shrink as the number of credit tranches rises.[2]

Ratability and the Valuation Space in Structured Finance

Every deal structure has a finite initial set around the solution vector within which the deal converges, another such set where it diverges, and a border where it neither converges nor diverges. To see this point, consider the geometry of mappings generally and consider that the deal's root locus behaves as a quadratic function.

$$a x^2 + b x + c = 0 \qquad 22.19$$

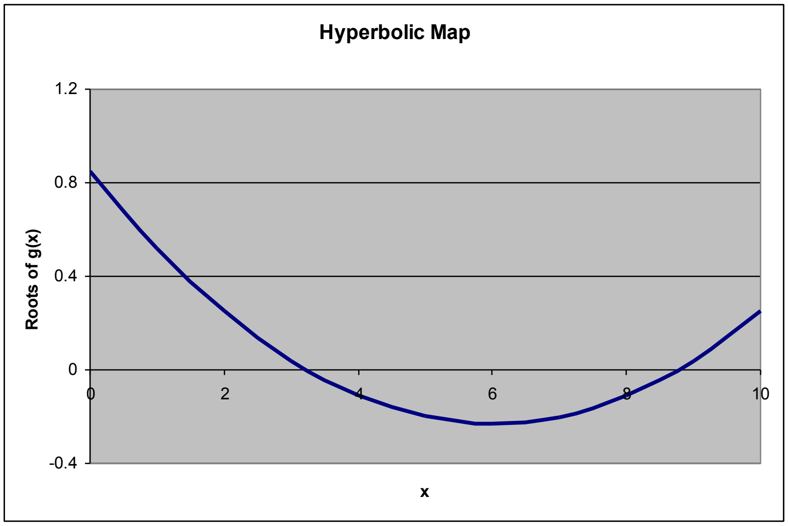
The roots of quadratic forms such as 22.19 are determined by their determinant D given by:

$$D = b^2 - 4ac \qquad 22.20$$

Depending on the value of D, there will be 2 real roots [$D > 0$], a single real root [$D = 0$] or two imaginary roots [$D < 0$]. Call *hyperbolic* situations those where $D > 0$ (the roots are *real* and *exist*), *parabolic* those where $D = 0$ and *elliptic* those where $D < 0$. Below, in Figure 22.3, the weights are $a = 0.03$, $b = -0.36$, $c = 0.85$. The determinant is $D = (0.36)^2 - 4(0.03)(0.85) = 0.276 > 0$ and the equation has two real roots. Had $D = 0$ prevailed, the graph would have barely touched the y axis. If $D < 0$ held, the whole curve would have been above the y axis and the roots, *imaginary*. This is also how the deal's valuation process presents itself, as a mapping function like that of $g(x)$.

[2] ***Elements-Chapter 22*** reviews the algebra of a contractive mapping in one dimension and details on the relationship between the mapping function slope and the convergence process.

Figure 22.3: The Concept of the Hyperbolic Map



Finding the roots of the deal is solving for the roots of $g(x)$, and the same nomenclature applies. When the roots of the deal are real, we refer to a hyperbolic iteration whereby the determinant of $g(x)$ is greater than zero. In Banach's theorem, this is $q < 1$ and a solution exists. Such deals are *well posed* since their mapping function will always lead to *Value*.

A well-posed deal does not mean riskless—it means ratable. "*Value*" implies existence and uniqueness. When the roots are the same, we have the parabolic situation and $D = 0$ holds, which corresponds to $q \approx 1$. A real solution tenuously exists—a *meta-stable* situation. When $D < 0$, the roots are imaginary and the deal will diverge. It does not work, or is *ill-posed*. Spectrum cannot rate a non-deal.

The topology of structured financial analysis contains a fuzzy region where the deal is neither strictly hyperbolic nor elliptic. Instead of a line, the parabolic region will be a strip and deals lying on it will move in and out of convergence due to the irreducible randomness and vagaries of MC analysis. Valuations launched inside the parabolic strip may very well converge in one MC simulation and diverge in the next one. This region is by far the most tempting deal-space areas for an arranger to obtain a rating on, just because its ratability is borderline.

<u>The Two Vector Spaces Inherent in Structured Finance</u>

Deal space really consists of two distinct sub-spaces: yield and parametric. In effect, valuation needs to achieve two distinct and independent forms of convergence. These two concepts address completely different aspects of the valuation puzzle and require separate discussions. But their topological properties are closely related, so understanding achieved in one space can be carried over to the other.

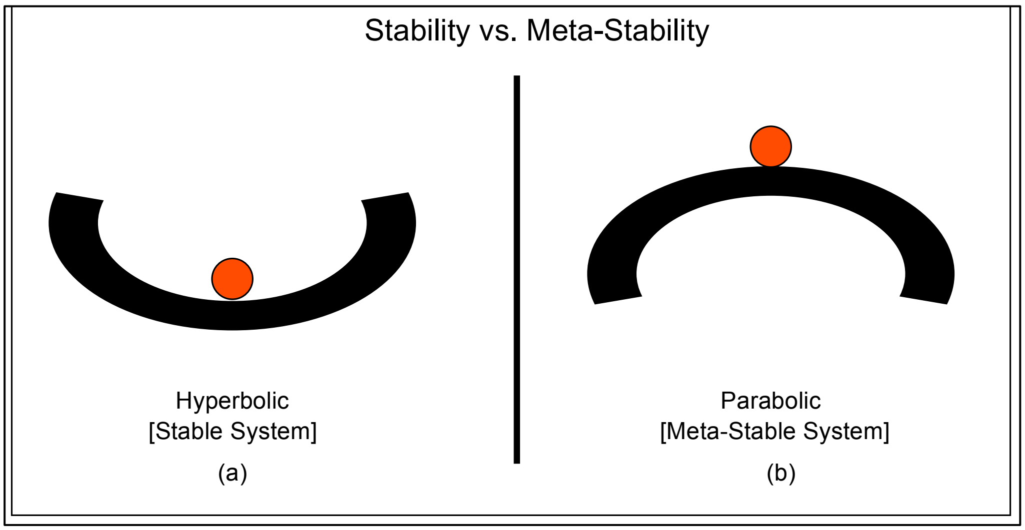
Yield Space
As its name indicates, yield space refers to the space of liability interest rates. It makes up the schema that enables us to talk about the <u>uniqueness</u> of structured valuation (*see Section 22.6 below*). This space is a natural outgrowth, generalization and consequence of Banach's fixed-point theorem.
Parametric Space
This more abstract space involves basic deal parameters, things like issuance levels, reserve accounts and delinquency triggers. It is within this space that the notion of "optimality" within structured analysis can be defined. As a result, it is just as important as yield space. More importantly, it is *freedom* space.

Meta-Stability in Structured Finance

Meta-stability is a concept borrowed from the field of dynamics and is significant to rating analysis. It refers to instability to relatively small disturbances. This is the case for non-investment grade bonds, whose payment certainty is vulnerable to multiple dimensions of risk. It also describes the tenuous convergence behavior of tranches subjected to randomness in Monte Carlo valuation techniques.

Consider Figure 22.4-a representing a truly stable environment while Figure 22.4-b tries to convey the physical and intuitive notion underlying meta-stability, where instability would result from any disturbance on the right or left. More complicated situations, called saddle points, exist in structured valuations, where some regions are stable and some non-stable. Non-linear space is intricate.

Figure 22.4: The Distinct but Related Notions of Stability and Meta-Stability



VI. Convergence in Yield Space: Banach Land

Under Banach's thought leadership we can define the deal's existence as a region wherein an iterated sequence of tranche rates became a Cauchy sequence. This definition allows us to define unambiguously what a deal means. Consider Figure 22.5 as a yield-space schematic for a two-tranche transaction. We define the rate vector as r_1 for Class A and r_2 for Class B. The dark spot in the lightly shaded portion is the solution rate-vector r_∞. The convergence regions are non-overlapping. A parabolic region would be much thinner than shown here.

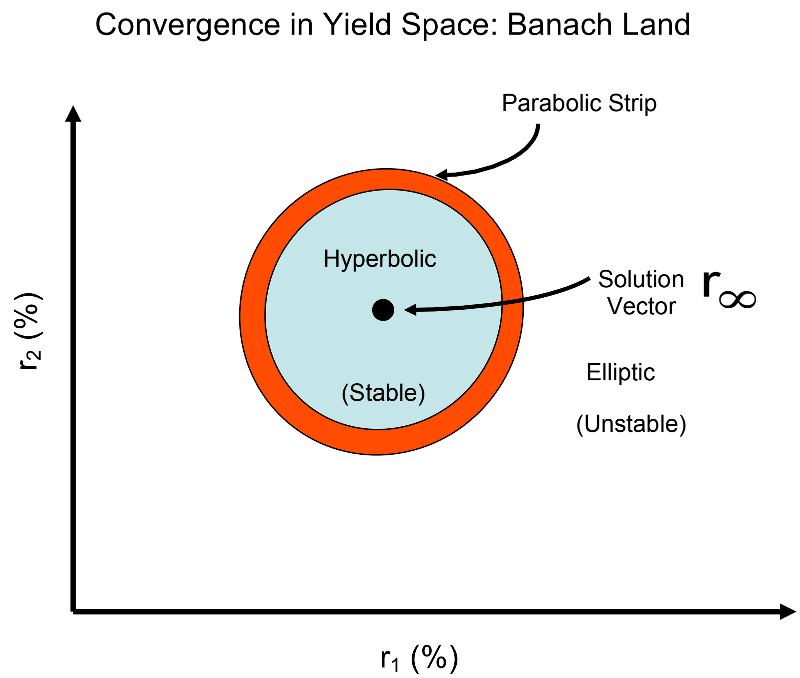
Given any starting rate-vector in the hyperbolic region, iteration of the deal's mapping function $g(r)$ converges to the unique fixed point in Figure 22.5. The hyperbolic region does not include zero interest rates. In theory the width of the parabolic strip can be made smaller by increasing the number of Monte Carlo scenarios, but it is fundamentally constrained by the basic uncertainty inherent in issuer data and by the limited resolution compatible with the capital markets' measure of value, whichever is smallest.

One cannot ask the simulation to be more precise than the resolution found either in the original data or in our own limits of measurability. Even on a theoretical basis, the width of the parabolic strip will always remain finite. In practice, the maximum number of scenarios N_M can be set equal to the minimum value such that an additional scenario would change the overall deal average reduction of yield by at most 1/100th of a basis point:

$$N_M = \min_N \left\{ \left| \Delta IRR(N+1) - \Delta IRR(N) \right| \le 0.01 \right\} \qquad 22.21$$

This definition is arbitrary, but it is Spectrum's cut-off point.

Figure 22.5: Convergence in Yield Space



The Value of the Deal

Spectrum's definition of the *Value* of a deal V in formal and practical, non-ambiguous terms, is the unique solution-vector r_∞ located in the hyperbolic region:

$$r_\infty \equiv r \,|\, \left\| \frac{g(r) - r}{r} \right\| \leq \varepsilon \qquad 22.22$$

$$V \equiv r_\infty \qquad 22.23$$

In equation 22.22, the error bound ε should be set somewhere around 1%. Randomness will usually prevent ε from being set too close to zero.

As dictated by Banach's theorem, equation 22.22 is a practical definition of when to quit when attempting to reach the fixed point. As already pointed out, statistical error and other constraints will always prevent us from getting infinitely close to the asymptotic limit implied by equation 22.6 and we will have to remain content with the stop law measured by ε. However, the deal's basic *Value* range thereby implied is not something "wrong" with this method. Knowledge has limits. On the other hand, what *is* wrong is for a ratings agency to pretend to offer perfect knowledge.

Calibration

Strictly speaking, the *Value* of the deal refers not to an outcome but to a process. Deal valuation is the re-enactment of this process, not the computation of a number. In practice, one obviously needs produce one, but as far as pricing goes, a wide array of choices will work just as well in allowing markets to clear. To assign a numerical figure to *Value* is to *linearize* the deal. However, because the latter concept is so ambiguous, we use the word *calibration*. It refers to the determination of the basis on which a price will be computed.

Likewise, any freshman chemistry student knows a thermometer needs to be calibrated before using. In the end, it does not matter whether water freezes at zero or thirty-two degrees, for what really matters is that we are all using water. The world of chemistry looks the same either way, and so does the world of finance. Assuming the deal will be sold at that price, the latter becomes its *fair* value. The word *fair* means that price and *Value* are the same number, and nothing else. When this is done and widely agreed upon, we refer to the associated figure as the *fair market value*.

V. Convergence in Parametric Space and Deal Optimality

A parametric space delineates the multi-dimensional manifold formed by the deal's basic parameters. Although unlimited theoretically, there are seldom more than ten arbitrary parameters in any realistic transaction. What follows is a short account of convergence properties inside parametric space.

<u>The Topology of Parametric Space</u>

Convergence to a fixed point in yield space is the process of achieving self-consistency in structured analysis. A fixed point can only be "fixed" for given structures. Only when basic deal parameters, tranche sizing, trigger levels, etc., remain unchanged can we truthfully claim a fixed point exists and is unique.

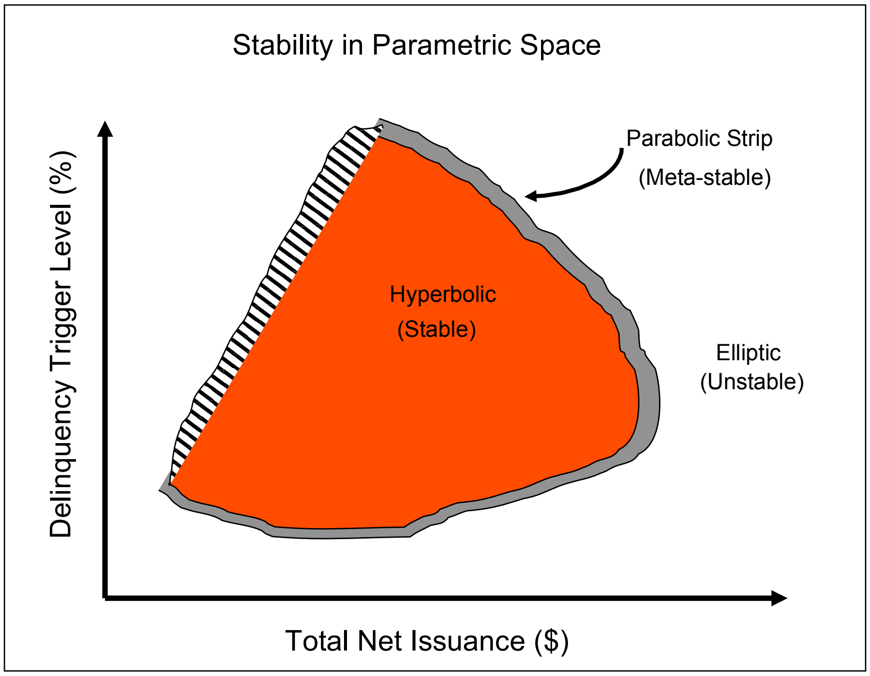
What happens to the fixed point if structural parameters are modified? It stands to reason the deal will not work at arbitrary issuance levels no matter what others may say about it. Otherwise, what prevents indefinite issuance? Deals will not possess *Value* outside a finite region of parametric space, similar to yield space. Inside this stable region, the deal will converge to a different fixed point, in theory at least, with respect to each parametric combination. Outside the stable region, it will diverge. An intermediate region will also exist that is meta-stable. The intuition garnered from yield space can therefore be carried over unaltered to parametric space where we observe the same three convergence regions.

By analogy with yield space, we designate convergence regions in parametric space using the labels hyperbolic, elliptic and parabolic respectively for the same deal, where they are the axes along which we measure its parameters. Think of a deal's parameters as the values a, b and c on the right-hand side of 22.19. Consider Figure 22.6 below, which is nothing but the analogue of Figure 22.5 in parametric space. It shows what happens topologically inside a deal where we have conceptually reduced the number of degrees of freedom to two, here arbitrarily chosen as <u>Total Net Issuance[3]</u> and <u>Delinquency Trigger Level</u> as a percentage of the currently outstanding pool balance.

Just like Figure 22.5, note how the hyperbolic region in Figure 22.6 is finite. For Total Net Issuance, it is clear we can issue a zero aggregate amount of securities out of any pool and this 'solution' is feasible and stable in the trivial sense. A parabolic strip can be found around the hyperbolic stability region, with the range of allowable trigger levels compatible with feasibility decreasing as issuance increases. The generally negatively sloped orientation of the stable region reflects that as total issuance increases, the range of delinquency trigger levels leading to convergence must decrease since more spread must be captured inside the structure to justify the same rating, or simply to allow the deal to converge at all.

[3] This can be defines the aggregate principal balance of bonds issued less any initial deposit into a spread or reserve account.

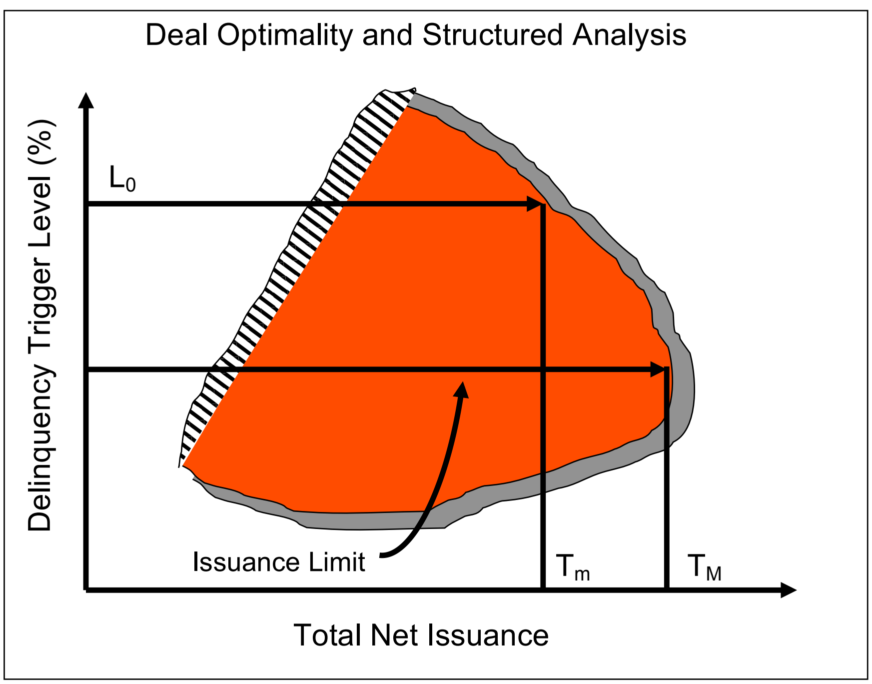
Figure 22.6: The Parametric Space and Deal Pricing



Towards a Definition of the "Optimal" Deal

Now consider Figure 22.7 below showing the situation, as it is likely to present itself after mapping out the hyperbolic manifold. In practice the space will be multi-dimensional, not this intuitive or smooth.

Figure 22.7: Deal Optimality and Structured Analysis



As trigger levels increase, Total Net Issuance T compatible with a convergent solution decreases, because less spread can be accumulated inside the structure under stress cases.

Here, a one-parameter family of solutions to the liquidity problem exists—those extending from zero to the Issuance Limit line. This family represents the deal's corresponding issuance window and can be defined as the range $T \in [0, T_M]$ whereby T_M is the maximum feasible issuance. Given trigger level L_0 this seller can only issue an amount T_m, where $T_m < T_M$. The parabolic strip's entire inner border should be considered the locus of optimality for this transaction, and not simply the point T_M. The issuer and the investors can be left to decide where along that border the deal should go to market.

Finally, all deals are optimal with respect to some constraint. If there are *ad hoc* restrictions on credit ratings, a lesser issuance level than what is economically optimal will be adopted. For any given deal, there is an infinite array of structuring possibilities, each offering its own brand of optimality. The only constant in all of this is that structured finance valuation in that starts with the goal and omits the process is technically a non-starter and operationally destabilizing to debt capital markets.

VI. A Live Example of Deal Convergence in Yield Space

As an example of convergence in an actual transaction, Table 22.2 and Figure 22.2 below shows the target deal's convergence history in yield space for the parametric set displayed in Table 22.1. *Alpha* in Table 22.2 is the convergence criterion, here set to 10 bps. We illustrate coding the non-linear valuation loop in the VBA implementation that led to Table 22.2, but without relaxation mechanics.

Table 22.1: Parametric Set for the Target Deal

Parameter ID	**Value**
Asset Class	Autos
Principal Allocation Scheme	*Pro Rata*
A Class Initial Balance	\$29,937,671.49
B Class Initial Balance	\$7,484,417.87
Delinquency Trigger Type	Binary
Delinquency Trigger Index	Quart. 60+ DPD
Delinquency Trigger Level	10%
Reserve Account Target Level (% Current Bal)	3.0%
Pool Balance	\$37,422,089.37
Estimated Recoveries	See Chapter 17
Percent Current Loans at Closing	100%
Gross Expected Loss	17.37%
Macroeconomic Simulation Method	Clayton Copula
Estimated Recovery Delay	3 months
WAC	16.65%
WAM	60.34 months

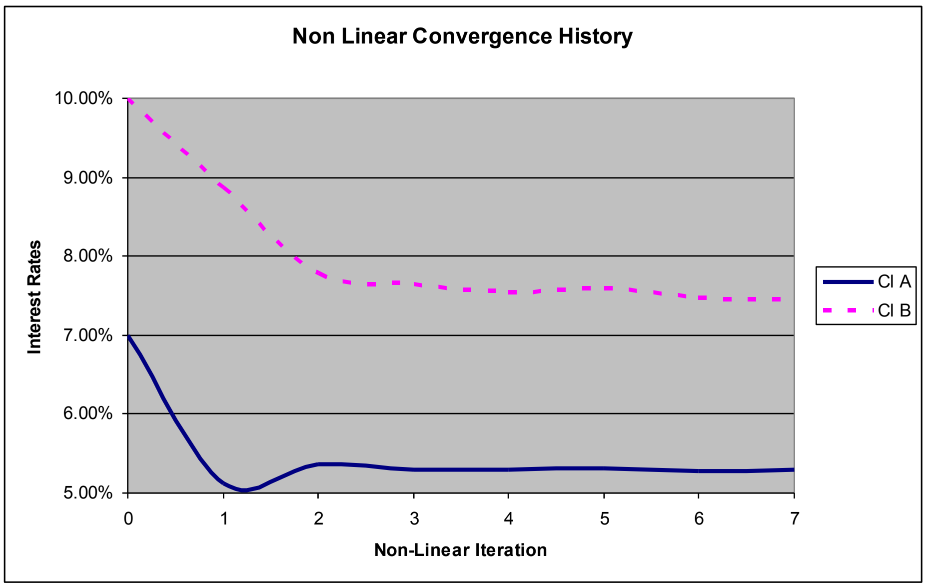
With a 10 bps threshold value for Alpha, the rates converge self-consistently to within 1 bps, which is far more accurate than any current or past method of analysis. Equilibrium interest rates are given on the penultimate row (Run ID: 7) of Table 22.2, to match our ratings of Aa1 for Class A and Ba1 for Class B.

Finally, as an example of coding the non-linear valuation loop, we present as Exhibit 22.1 the VBA implementation that led to Table 22.2, but without relaxation mechanics, as in the follow-on section.

Table 22.2: Target Deal Convergence History in Yield Space

Run ID	A DIRR (bps)	B DIRR (bps)	A Rate (%)	B Rate (%)	Alpha
0	-	-	7.00	10.00	1.0
1	0.97	129.98	5.12	8.86	0.2375
2	0.39	62.06	5.36	7.78	0.0623
3	0.33	66.32	5.30	7.63	0.0137
4	0.28	62.45	5.29	7.54	0.0028
5	0.32	66.47	5.31	7.58	0.0028
6	0.26	58.38	5.28	7.47	0.0061
7	0.26	58.97	5.29	7.46	0.0010
Class A WAL: 27.5 months			Class B WAL: 28.0 months		

Figure 22.2: Non-Linear Convergence History in Yield Space



Run the nonlinear loop

Exhibit 22.1: Non-Linear Convergence in Yield Space

```
Sub AnalyzeDeal()
Dim deal As deal, n As Integer, alpha As Double, r As Range
Call ReadDeal(deal)
deal.enablePrepayModel = True
randVdc_seq = 1
Dim aInitBal As Double, bInitBal As Double
aInitBal = deal.alpha * deal.poolBal
bInitBal = (1 - deal.alpha) * deal.poolBal
Set r = Range("NonLinearConv")
r.ClearContents
Range("Ratings").ClearContents
Dim i As Integer
```

```
For i = 1 To max_non_linear_iterations
Dim ayr As Double, byr As Double, ata As Double, bta As Double, ar As Double, br As Double
Call RunDealSim(deal, n, ayr, byr, ata, bta)
ar = YieldCurvePlusSpread(deal, ata, ayr)
br = YieldCurvePlusSpread(deal, bta, byr)
alpha = (aInitBal * Abs((ar - deal.aRate) / deal.aRate) + bInitBal * Abs((br - deal.bRate) / deal.bRate)) /
deal.poolBal
r.Cells(i, 1) = ayr
r.Cells(i, 2) = byr
r.Cells(i, 3) = ar
r.Cells(i, 4) = br
r.Cells(i, 5) = alpha
'At the convergence threshold, exit the loop
If alpha < threshold Then Exit For
End If
deal.aRate = ar
deal.bRate = br
Next i
'Obtain Spectrum Grade Credit Ratings
Range("Ratings").Cells(1) = GetRating(ayr)
Range("Ratings").Cells(2) = GetRating(byr)
End Sub
```

<u>Successive Relaxation in Yield Space</u>

Spectrum applies the successive relaxation technique to allow tranche-wise convergence within yield-space using a scaling factor appropriate to each class. This is not the basis for a theory of relaxation, it is only a rule of thumb. To implement relaxation, proceed as follows:

(1) Assume that in the target deal we perform Monte Carlo simulation n using yield-vector r^n and are now ready to compute the next iterate-vector r^{n+1} using a yield update rule formulated as follows:

$$r_i^{n+1} = r_i^n + \lambda_i \Delta r_i^n \qquad 22.24$$

(2) λ_i is a tranche wise relaxation coefficient of order one. It rules the relaxation process *via* the tranche increment Δr_i^n computed *via* the yield curve function Y_c as follows:

$$\Delta r_i^n = Y_c(\overline{t_i^n}, \Delta IRR_i^n) - r_i^n \qquad 22.25$$

(3) The yield curve model (*next Section*) takes as its inputs, with respect to tranche i, average life $\overline{t_i^n}$ and average yield reduction ΔIRR_i^n, both of which are outputs of any given MC simulation. If $\lambda_i = 1$ the new rates are simply set equal to the output of the yield curve model (i.e. the abscissa of the Y_c function) resulting from the last MC simulation.

(4) Now formally define the familiar notions of successive over- and under-relaxation, SOR and SUR respectively, as follows:

$$SOR \equiv \lambda_i \in (1, \infty) \qquad 22.26$$

$$SUR \equiv \lambda_i \in [0, 1) \qquad 22.27$$

Whenever $\lambda > 1$, tranche rates will be updated by a greater amount (+ or -) than warranted by the yield curve transformation, while when $\lambda < 1$ holds, the opposite will be the case. Over-relaxation drives the system to a fixed-point solution faster. Under-relaxation, the reverse, dampens yield oscillations that could cause provisional solutions to spin out of control when a *bona fide* fixed-point solution does exist.

To reach the solution rate-vector as fast as possible, over-relaxation is better as long as it does not destabilize the deal. Usually it is better to drive the provisional solution vector initially to its fixed point using over-relaxation on senior tranches and under-relaxation on junior tranches. It can also be helpful to modulate tranche-wise relaxation factors in a cascading manner. Modulation matters in managing the runtime of deals with many tranches. For 5+, a properly designed tranche-wise relaxation plan can yield significant CPU-time savings.

A good relaxation-factor modulation index is the slope of the yield curve update Δr_i^n as a function of the iterate counter n. The value of tranche-wise relaxation parameters λ_i should be specified as a negatively sloped, monotonic function of Δr_i^n, as shown in Figure 22.7 below.

Figure 22.7: Suggested Form for the λ_i Modulating Function

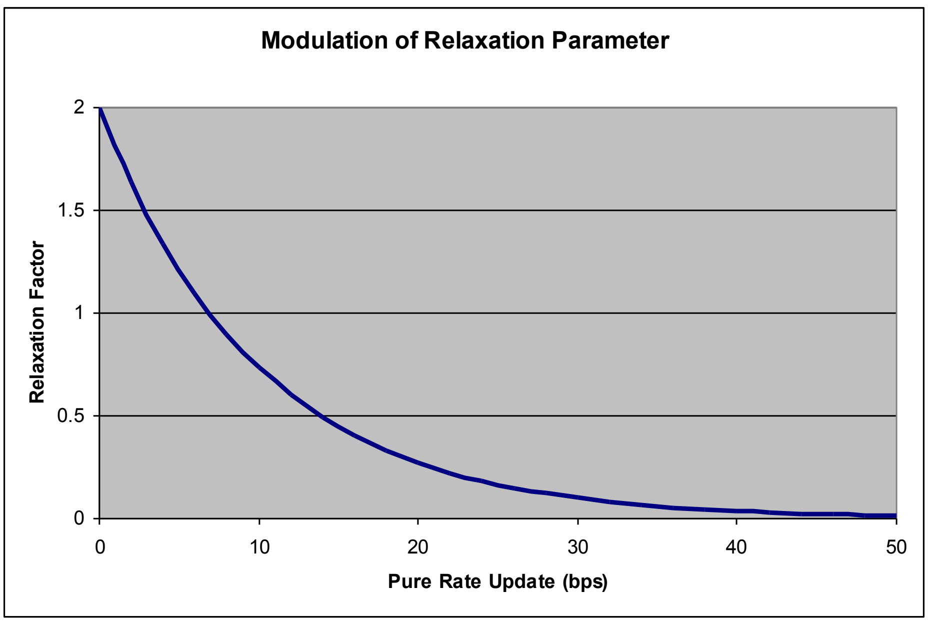


Exhibit 22.2: The Successive Over-Relaxation Algorithm inside the Non-Linear Loop

```
- - - - - - (same non-linear code as before) - - - - - - - - -
br = YieldCurvePlusSpread(deal, bta, byr)
ar = deal.aRate + deal.aRelax * (ar - deal.aRate)  (Class A relaxation)
br = deal.bRate + deal.bRelax * (br - deal.bRate) (Class B relaxation)
alpha = (aInitBal * Abs((ar - deal.aRate) / deal.aRate) + bInitBal * Abs((br - deal.bRate) / deal.bRate)) / 
deal.poolBal
- - - - - - (same non-linear code as before) - - - - - - - - -
```

VII Yield Curve Modeling

At the outset, remember that no yield curve *model* needs to be implemented in practice since the yield curve itself is observable. Instead, a simple polynomial form will do, fitted to the daily values found in widely circulated publications. However, the valuation problem is sufficiently non-linear on its own without adding the additional noise arising from the vagaries of the marketplace (causing discontinuous yield curves to emerge). To bootstrap our model Spectrum uses, and recommends the use of, a smooth well behaved yield curve model. We discourage attempts to ape precisely empirical yield curves, for they will be all wrong tomorrow anyway.

Recall that total yield can be conceived as the sum of contributions from three distinct sources:

- The equilibrium price of instantaneous transfers (risk-free rate)
- The credit risk premium
- The liquidity premium

<u>The Basic Yield Curve Model</u>

Following the above considerations, we can formally state our yield curve model's basic schema:

Yield Curve = Treasury Yield + Credit Spread

This relation can be formalized using the same basic continuous functions we used in Chapter 21 of "*The Analysis of Structured Securities*" where we dealt with the CDO of ABS. We reiterate the simple diffusion model here:

$$Y_c(\bar{t}, \Delta IRR) = f(\bar{t}) + \alpha\sqrt{\bar{t}\,\Delta IRR} \qquad 22.39$$

In equation 22.39, the first term is the risk-free rate and the second term is the diffusive credit spread term with the familiar square root behavior. For the risk-free rate functional form $f(t)$, we had selected our old friend the logistic curve, but this time equipped with carefully chosen parameters:

$$f(\bar{t}) = \frac{r_\infty}{1+\beta e^{-\delta \bar{t}}} \qquad 22.40$$

In equations 22.39 and 22.40 the input and output variables were defined as shown in Table 22.2 below.

Table 22.2: Variable Definitions for the Basic Yield Curve Model

Variable	Definition
Y_c	Total rate as the sum of the risk-free rate and the credit spread (%)
$\bar{t}$	The average life of a tranche (years)
ΔIRR	The average reduction of yield on a tranche expressed as a percentage
α	Calibration multiplier
r_∞	Limiting value of the risk-free rate as $\bar{t} \to \infty$
β	Logistic curve (equation 22.40) shifting parameter
δ	Logistic curve (equation 22.40) spreading parameter (year)$^{-1}$

Using the Spectrum or Moody's rating definition of average reduction of yield, total yield Y_c is a function of broad rating category using this parametric set in Figure 22.8: $\alpha = 0.02$, $r_\infty = 0.08$, $\beta = 0.9$, $\delta = 0.21$.

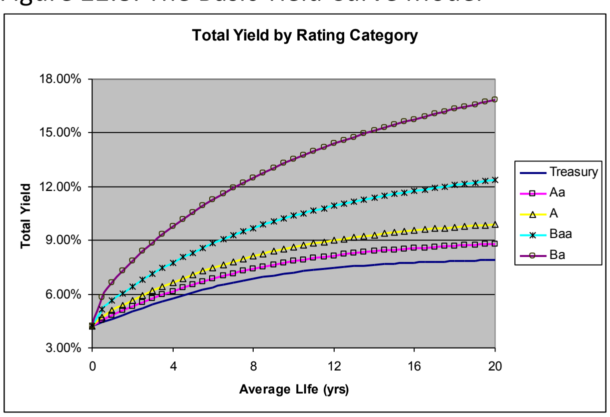
The shaded area below shows a VBA implementation of the same algorithm, where the yield curve is integrated with the non-linear portion of deal analysis:

Exhibit 22.3: Yield Curve Model Implementation

```
Sub ReadYieldCurveModel(deal As deal)
Dim r As Range
Set r = Range("YieldCurveModel")
deal.ycRInf = r.Cells(1)
deal.ycAlpha = r.Cells(2)
deal.ycBeta = r.Cells(3)
deal.ycDelta = r.Cells(4)
End Sub

Function YieldCurvePlusSpread(deal As deal, ta As Double, yr As Double) As Double

YieldCurvePlusSpread = ((deal.ycRInf * 100) / (1 + deal.ycBeta * Exp(-deal.ycDelta * (ta / 12))) +
deal.ycAlpha * Sqr((ta / 12) * yr / 100)) / 100

End Function
```

Spectrum uses a model fitted to actual market data rather than an empirical curve. An analytical model does not have the quirks common to empirical yield curves and is more intuitive; fitting should guarantee “smoothness.” Deal discontinuities will tend to arise because of certain structural features that kick in under specified conditions.

Figure 22.8: The Basic Yield Curve Model



VIII. Annotated Biography of Banach and Proof of the Banach Fixed Point Theorem

In Memoriam

Stefan Banach (188?-1945)

Stefan Banach, one of the 20th Century's leading mathematicians, was the illegitimate son of Stefan Greczek, a local tax official who was not married to Banach's biological mother, who vanished shortly after the boy's baptism. On his birth certificate, she was listed as Katarzyna Banach, who is believed to have been either his real mother's servant or someone who took care of him as an infant. Later on, when Stefan Jr. tried to find out the truth about his natural mother, his father told him he had been sworn to secrecy. Thus with Banach, the search for his inner self took on a very personal dimension.

Young Stefan was first brought up by his grandmother but later lived with Franciszka Plowa, who lived in Krakow with her daughter Maria, whose guardian, the public intellectual Juliusz Mien, taught Stefan to speak French, respect education, and who recognized immediately Banach's mathematical talent. In high school, Banach was considered mediocre. Upon graduation, convinced there was nothing left to prove in mathematics, he studied engineering at Technical University at Lvov until 1914.

The event that changed Banach's life forever was a 1916 chance encounter with Hugo Steinhaus, a leading Polish mathematician who was about to start teaching at Lvov. Steinhaus told Banach about a problem he had been struggling with. Banach found a solution in a few days. They co-wrote a paper published in 1918; and Banach started his academic career in Lvov as an assistant to Lomnicki and later submitted a doctoral dissertation entitled "*On Operations with Abstract Sets and their Applications to Integral Equations,*" a work now believed to mark the birth of functional analysis.

Although this path to a doctorate was irregular, Banach having no mathematics qualifications, an exception was made and he obtained the Ph.D. degree. In 1922, he presented his *Habilitationsschrift* on measure theory. This is the post-doctoral thesis that allows someone to teach at a University under the German system. Banach was now a professor of mathematics at the Jan Kazimierz University in Lvov.

In the inter-war years, Banach and Steinhaus remained close collaborators. In 1931, they started editing a new series of mathematical monographs from Lvov while colleagues based in Warsaw were also contributing material. The first of these tomes, written in Polish by Banach himself in 1931, quickly became a classic after its publication in French ca. 1932 under the title "*Théorie des Opérations Linéaires*"—theory of linear operations. And the rest is part of the modern history of mathematics. Banach is credited with the foundation of functional analysis and the axiomatic definition of the "Banach Space" (coined by Fréchet), which is central to structured financial analysis. He made major contributions to the theory of topological vector spaces, measure theory, integration and orthogonal series. He also proved a number of fundamental theorems, notably the Hahn-Banach theorem, the Banach-Steinhaus theorem, the Banach fixed-point theorem (see below), the Banach-Alaoglu theorem and the well-known Banach-Tarski decomposition theorem. In sum, Stefan Banach was a brilliant mind and a great man who laid the theoretical foundations of structured finance.

***

Spectrum goes through the proof of Banach's fixed-point theorem below to provide a deeper, more rigorous mathematical foundation to structured finance than it currently has. In our view, it is not possible to learn structured security valuation without comprehending or proving it.

In finance, analytical results are so rare that we should celebrate any time we come across something that holds water, and we do. The canonical statement of Banach's fixed-point theorem runs as follows:

Premise: Let (X,d) be a non-empty, complete metric space and let T be a contraction mapping on (X,d) with constant q. Choose an arbitrary $x_0 \in X$ and define the sequence $\left(x_n\right)_{n=0}^{\infty}$ by $x_n = T^n x_0$. Further, let $a \equiv d(Tx_0, x_0)$.

Conclusion: Then, it follows that T has a unique fixed point in X.

Metric Space

A metric space is a vector space equipped with a metric. For instance, two-dimensional Euclidean space is such an animal, but the Pressure-Volume-Temperature space of thermodynamics is not. Obviously, when you define a metric space, it is a good idea to define the metric as well. In a two-dimensional, Euclidean space for example, the metric d with respect to the two points P_1 and P_2 with coordinates $(x_i, y_i), i \in [1,2]$ would be defined as follows:

$$d \equiv \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The above metric looks suspiciously like the distance measure in plane geometry, which is why we chose the letter d to label it. In the vast majority of cases, the metric d applied to two points will be the Euclidean distance between them.

Complete Metric Space

A complete metric space is one in which all Cauchy sequences converge to a limit. This assumption is critical to prove Banach's theorem. In a Cauchy sequence, the successive elements $a_n, n \in [0, \infty)$ taken as a pair become increasingly close to each other. With respect to the above metric space with metric d, a Cauchy sequence can be defined formally as follows:

$$\lim_{\min(m,n) \to \infty} d(a_m, a_n) \to 0$$

This definition is more general than the relationship to deal valuation, but the intent is identical.

Contraction Mapping

A contraction mapping, also called a *hyperbolic* mapping, is characterized by the fact that the mapping of any two points in X will result in a reduction, or *contraction*, of the distance between their images compared to that between the original points. If x and y are chosen arbitrarily in X, then for T to be a contraction mapping in X the following condition must hold for some scalar q with $0 \le q < 1$:

$d(Tx, Ty) \le q\, d(x, y)$ (cf. inequality 22.7)

The scalar q needs to be the largest value over X that can be found, since the requirement is for the ratio of the distances between the mapped and unmapped points to always be less than q. Also, q cannot be *equal* to unity, as this would simply move some of the points around the space without changing the distance between them. In effect, if this held everywhere in X we would have a simple rotation followed by a potential shifting of the origin. The parameter a is the distance between the first two iterates in the sequence, i.e. $a \equiv d(x_1, x_0)$.

Proof of Banach's Fixed Point Theorem

First, show by mathematical induction that for any $n \ge 0$, we have:

$$d(T^n x_0, x_0) \le \frac{1 - q^n}{1 - q} a \qquad (1)$$

For $n = 0$, the result is obvious. For $n \geq 1$, suppose $d(T^{n-1}x_0, x_0) \leq \frac{1-q^{n-1}}{1-q}a$ holds, and it does so trivially for $n = 1$. Then, we have from the assumption of the contraction mapping:

$$d(T^n x_0, x_0) \leq d(T^n x_0, T^{n-1}x_0) + d(T^{n-1}x_0, x_0) \leq q^{n-1}\, d(Tx_0, x_0) + \frac{1-q^{n-1}}{1-q}a = \frac{q^{n-1}-q^n}{1-q}a + \frac{1-q^{n-1}}{1-q}a = \frac{1-q^n}{1-q}a$$

Hence, we have proved inequality (1) by using the triangle inequality (the sum of the length of two sides is always greater than the length of the third) and repeated application of the contractive property $d(Tx, Ty) \leq q\, d(x, y)$ that holds for T. By induction, the property holds for $n \geq 0$.

The next part of the proof consists in showing that the elements $T^n x_0, n \in [1, \infty)$ of the sequence starting at x_0 form a Cauchy sequence, which means that successive elements get increasingly close to each other. Once this is done, we can conclude that the sequence has a limit since the metric space is assumed complete. As you have already guessed, this limit will turn out to be a fixed point of operator T.

Given an $\varepsilon > 0$, we can find an integer N such that $\frac{q^n}{1-q}a < \varepsilon, \forall\, n \geq N$ because the ratio $\frac{q^n}{1-q}a \to 0$ as $n \to \infty$. Now, for any pair m, n (and assuming, without loss of generality, that $m \geq n$), we have:

$$d(x_m, x_n) = d(T^m x_0, T^n x_0) \leq q^n\, d(T^{m-n}x_0, x_0) \leq q^n \frac{1-q^{m-n}}{1-q}a \leq \frac{q^n}{1-q}a < \varepsilon$$

The sequence is a Cauchy sequence so by the theorem's assumptions, it possesses a limit point, say x^*. Next, show that x^* is a fixed point of the operator T. This is done by *reductio ad absurdum*.

Suppose that x^* is not a fixed point of the operator T, then by definition we have $\delta \equiv d(Tx^*, x^*) > 0$. Because x_n converges to x^*, there is an integer N such that $d(x_n, x^*) < \delta/2$, $\forall\, n \geq N$. Playing the same game as before (triangle inequality), we find:

$$d(Tx^*, x^*) \leq d(Tx^*, x_{N+1}) + d(x^*, x_{N+1}) \leq q\, d(x_N, x^*) + d(x^*, x_{N+1}) < \delta/2 + \delta/2 = \delta$$

So, the condition $d(Tx^*, x^*) = 0$ must hold. x^* is a fixed point of the operator T inside metric space X.

The last part of the proof is to show that the fixed point x^* is unique. To do this, we again have recourse to the negative method. To wit, suppose there exists another fixed point x' of T in X and that $x' \neq x^*$. This means $d(x', x^*) > 0$, but then: $d(x', x^*) = d(Tx', Tx^*) \leq q\, d(x', x^*) < d(x', x^*)$

This is clearly a contradiction. If two fixed points exist, they must in fact be the same point.

Relationship to Linear Iterative Sequences

This discussion is related to Spectrum's educational material on linear operators in eigenvalues. Eigenvalues can be conceived as shrinking or stretching factors that modify input vectors upon the operation of a matrix. An analogy exists between the linear, matrix operators we discussed previously and the hyperbolic, non-linear operators at the heart of Banach's proof.

In effect, the parameter q above is the equivalent of the stretching or shrinking factor, and thus can be regarded as the "eigenvalue" of operator T. Since q is always less than one, the iteration of T leads to an "equilibrium" solution here as well. As before, the equilibrium solution is reached asymptotically in the limit $n \to \infty$.

Note also that the properties of the fixed point are transcendental, which means that as long as we are not quite at the fixed point x^*, the operator keeps the process moving ever so slightly. It is only at the precise equilibrium location that the operator simply reproduces its input.

Thus in practice the fixed point is an ideal limit that is never truly reached. As we approach equilibrium, spurious elements tend to vanish (cf. Markov chain eigenvalues less than unity in absolute value) and the dominant behavior is recognized as that corresponding to the leading eigenvalue.

A situation can also be imagined whereby $q > 1$ holds and a fixed point still exists; but unless the iterative sequence begins precisely there, by sheer luck, the system will never approach its fixed point on its own and will generally diverge. Thus, there is a close relationship between eigenvalue analysis and the discovery of fixed points within non-linear systems.

End: Spectrum's Primary Market Structured Finance Valution Method Supporting Its Rating Approaches

Beginning: Spectrum's Secondary Market Structured Finance Valution Method

I. Introduction

Spectrum has a unified approach for rating structured securities in the secondary market. It applies globally, across and down the credit spectrum. The basis of our approach is encoded in a patented utility process designed by the founders of Credit Spectrum Corp: ABSTRAK(R).

The conceptual basis is a re-valuation of pools of financial assets from underlying updated primary data elements obtained from transaction-specific servicer reports in a Monte Carlo framework. The outputs are fed to a logical inference engine that values the associated tranches using the cash flows, by truing up the fair market value of outstanding ABS, RMBS and CMBS transactions.

CDO of ABS require a further abstraction over the single-transaction ABSTRAK™ engine but becomes trivial after the time-series of tranche-specific cash flows become available.

<u>ABSTRAK(R) at a Glance</u>

The complete ABSTRAK(R) platform consists of two distinct software environments:

(1) The asset-side cash flow engine described further below
(2) The deal structuring tool known as the Waterfall Editor™ [WFE]

Spectrum's WFE© allows users to represent with infinite precision the allocation rules in a prospectus or PPM. It can be used in two independent modes:

(1) As a stand-alone application it enables the user to enter and test tranche cash-flow allocation rules (waterfall);
(2) Integrated with the ABSTRAK(R) it enables securities associated with an existing transaction to be valued.

II. The ABSTRAK™ Formalism

The valuation of structured securities inside ABSTRAK™ is a two-step process. The first step [**Calibration**] is *automated* and the second [**Monitoring**] is *automatic*.

<u>Step 1: Calibration</u>

Calibration refers to the *a priori* normalization of a transaction, i.e. its valuation founded on a universal basis. If they stem from a widely agreed upon basis, such calibrated values lie beyond logic. The same is true of the temperature scale for instance. The fact that the physical distance between the freezing and boiling points of water is partitioned into 100 degrees Centigrade is not a logical determination, only a generally acceptable and arbitrary assumption. In fact, the basis of temperature is itself irrelevant to the proper operation of chemistry if everyone can communicate findings based on it, hence the label 'basis'. It is in this way that water is the ground of the temperature scale. However, glycerin or honey would do just as well.

Basis Selection

Structured finance has a critical need for a widely acceptable basis on which to renormalize tranche valuations in the secondary market. Credit Spectrum currently uses our credit rating on the senior tranche or tranches. This is a good standard: every deal has one.

Using a mapping table, Spectrum's letter-grade rating is mapped to an average yield reduction from the initial coupon promise for fixed-rate tranches or the spread on floating-rate bonds. Spectrum's Aaa structured rating maps to a maximum average yield reduction of 0.06 bps. Once the deal is calibrated, the remaining tranche ratings are naturally computed. ABSTRAK(R) always delivers consistent ratings with respect to all liability-tranches.

Preparing the Deal

Before a transaction can be calibrated, it must be input into ABSTRAK™ by parameterizing its assets and liabilities. Assets normally consist of a set of *a priori* independent loans, leases, or other financial contracts in which liability holders own an *undivided* interest.

All loan-level cash flow results are aggregated to the pool level before they are fed to the waterfall for further analysis. In contrast to primary market analysis, where loan-level data matter, for secondary market monitoring further granularity should not improve the quality of the secondary market credit analysis unless additional credit-sensitive data elements were made available thereby. Collecting loan-level data elements can impose a significant financial and administrative burden on servicers without necessarily adding more accuracy than the monthly data feedback loop from monthly remittance reports.

In all Spectrum approaches to plain-vanilla consumer and commercial default analysis, the underlying assets' evolving credit status is parameterized with non-stationary Markov transition matrices, and modulated monthly by the changing cumulative loss curve.

The transition matrix system includes a credit loss tracking parameter [Asymptote] and a single-month mortality prepayment multiplier [SMM]. Both are calibrated to unity at closing. During the monitoring phase they will be adjusted automatically using remittance-report feedback.

Asset-side cash flows are computed for each collection period and fed to the target transaction waterfall, to retire the liabilities. This sub-step requires a file describing the capital structure of the transaction and cash-flow allocation rules. Spectrum programs the cash-flow allocation rules into WFE©, which produces an XML file with detailed allocation rules in Java. A Cloud Version of WFE© does the same in Java Script.

Single Month Transition Matrix

		End of period Values		
		30-day	60-day	Prepay
Beginning of period Values	30-day	0.002	0.001	0.02
	60-day	--	0.4	0.01
	Prepay	--	--	--

Calibration

Banach's fixed-point theorem posits the existence of a unique solution to the valuation problem. Axiomatically, a unique, marginal asset default short-rate volatility compatible with the transaction senior tranche rating exists. The short-rate volatility is computed through a one-dimensional root-locus procedure. The calibration step adjusts the short-rate volatility embedded in the stochastic, cumulative default process to the asset pool. Convergence is guaranteed because this is the solution of one equation in one unknown variable. Initial credit ratings on all corresponding deal tranches are simultaneously derived and stored in memory as by-products of the calibration step.

The short-rate volatility modulates these monthly transition matrices, which are updated with periodic servicer data. Thus Spectrum directly simulates deal cash flows directly by its cash flow transfer function. In addition to tranche-wise average yield reductions and equivalent ratings, ABSTRAK(R) 17 other predictive metrics with respect to each tranche, including relative fair market value [RMFV] under credit and market risk, RMFV absent market risk, credit duration, credit convexity, ands the tranche-level fair-market CDS spread. Metrics are updated monthly in the re-valuation process carried out during the monitoring phase (see Step 2 below). At that point, the transaction is considered "live" and ready for monitoring.

Step 2: Monitoring

Spectrum monitors all live, secondary-market transactions with the payment frequency and produces time-series results up to the most recent distribution date. A plain vanilla monitoring phase does not require human intervention. Graphical time-series outputs are automatically produced for each input and modeled parameter. Transactions can be incorporated into the monitoring process after closing and re-analyzed if critical or value-sensitive data elements are re-stated. Spectrum keeps separate records of the original and remonitored results in database.

Transaction monitoring remains in place until the associated structured securities have all paid off, or else the deal has been restructured. Spectrum believes this is the only way a Bayesian estimate of future performance based on the real-time integration of a deal's empirical history can remain current and thus, serve as the rationale for the valuation of the associated liabilities. After approximately 18 to 24 months of seasoning, performing deals usually stabilize, resulting in reliable forward-looking metrics.

Logistical Implementation

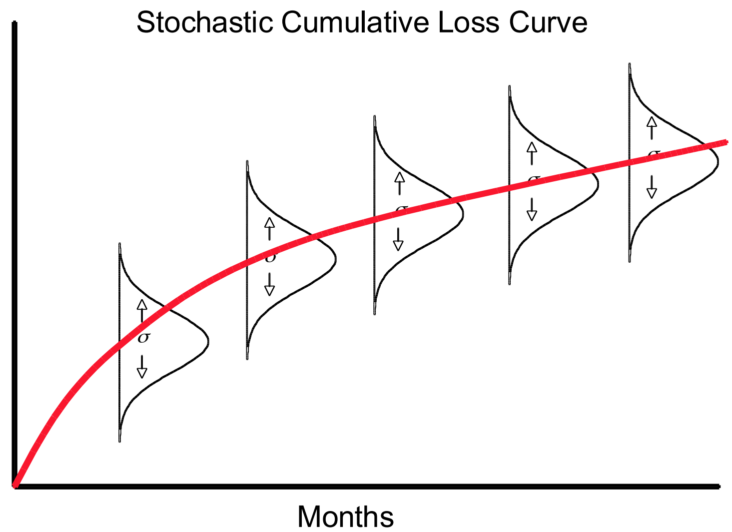
On a periodic basis, after transaction remittance reports have been produced and made available to ABSTRAK(R) the system automatically downloads updated XML- or spreadsheet-based deal files and runs the monitoring program. ABSTRAK(R) also downloads the reference yield curve and other ancillary time series data from various public websites, to update the ratings and ancillarymetrics. ABSTRAK(R) sends an email notice when the process is comoplete and the updates can be viewed, or alternatively data errors were found requiring investigation before a new monitoring attempt can be made.

Technical Implementation

At the technical level, the monitoring process is straightforward and proceeds through a four-dimensional, integral, root locus optimization algorithm very similar to the one used during calibration. Specifically, they are:

1) Prepayments
2) Defaults
3) 30-day dollar delinquencies, and
4) 60-day dollar delinquencies.

As a rule other delinquency-buckets are too unstable to deliver reliable signals. Otherwise, on a monthly basis, ABSTRAK(R) integrates the transaction's multi-dimensional cash flow density function as above and gets preliminary values for cumulative prepayments and cumulative gross defaults (graph below). The integration process is initially volatile but self-stabilizes post-closing in under (6) months to deliver smoothly varying time series with strong predictive power due to optionality and de-leveraging.



Despite the secular impact of asset-side stabilization just mentioned, the relative payment certainty of highly leveraged tranches, like the subordinated tranches of mortgage-backed security transactions, may remain relatively volatile throughout the transaction's lifetime. This should not be surprising, since the bottom 3% of the capital structure could be wiped out entirely by servicer information indicating a cumulative net loss rate of 3% or greater over the deal's remaining life. The increments in current marginal loss that would cause such an event, if extrapolated over the transaction's future, are small and counter-intuitive.

Spectrum compares preliminary ABSTRAK™-resident cumulative prepayment and default rates to empirically observed values for the same data elements, adjusting parameters of the status-transition process *in situ* to match empirical values. A small margin of error is allowed inside the matching algorithm, to allow for statistical errors and servicer data inconsistencies that always interfere with an otherwise purely analytical process. The above root-locus process is next applied to the first two delinquency buckets, after which the deal is ready for the monthly update process.

ABSTRAK(R) completes the monitoring step by carrying out a new Monte Carlo simulation from the next collection period until the remaining, weighted-average loan maturity [WARM] of the assets in the deal. This simulation uses the stochastic-process parameters that have been adjusted *via* servicer report feedback.

Because the monitoring phase begins with stochastic integration from closing to the current collection period, the associated re-valuation automatically takes into account the pool's current amortization as well as the deal's intricate liability pay-down schedule. ABSTRAK(R) thus can keep up in real time with transaction dynamics and incorporate the non-linear impact of credit losses and delinquencies on the fair market value of its liabilities, enabling investors and traders to witness in real time the tranches' unfolding credit improvement or deterioration.

For further details, please feel free to contact Sylvain Raynes at 1 212 867 5693 or sylvain@creditspectrum.com.